Working with Exponents and Logarithms

This handout is intended to serve as a refresher and reference for using exponential and logarithmic functions. We begin with an explanation of some of the basic properties and definitions associated with exponentiation and conclude with a list of useful properties of exponents and logarithms.

Exponentials

The value a^x is called "a to the power of x" and we say that x is the exponent of a^x. When n is a positive integer (one of the elements of the set $\{1, 2, 3, ...\}$), the value of a^n is defined as a times itself n times. In this case the value of a^{-n} is defined as 1 divided by the product of a times itself n times. Note that this only makes sense if you can divide by a. Consequently, expessions of the form 0^{-n} make no sense and are undefined when n is a positive integer. For nonzero a, $a^0 = 1$ by definition. Though 0^0 is technically undefined, some mathematicians think that it should equal 1 also. To recap, when n is an integer, and a nonzero, we have

$$a^n = \begin{cases} \underbrace{a \cdot a \cdots a \cdot a}_{n \text{ times}} & \text{if } n \text{ is positive} \\ \frac{1}{\underbrace{a \cdot a \cdots a \cdot a}_{|n| \text{ times}}} & \text{if } n \text{ is negative} \\ 1 & \text{if } n = 0 \end{cases}$$

One might be curious if the value of $a^x \cdot a^y$ can be simplified. As it turns out, $a^x \cdot a^y = a^{x+y}$. To illustrate why this is true when the exponents are integers, consider the following example

$$a^2 \cdot a^3 = (aa)(aaa) = aaaaa = a^5.$$

As another example, this one involving negative integers,

$$a^2 \cdot a^{-3} = (aa)(\frac{1}{aaa}) = \frac{aa}{aaa} = \frac{1}{a} = a^{-1}.$$

These examples fall short of proof since they only deal with specific values, but they should illustrate how the claim is true in general. This clearly works for any integers, and also works for real numbers in general, though it is more difficult to show that. Also note that from this we can show that $a^n \cdot a^{-n} = 1$ for all nonzero a and all n.

What about values such as $(a^n)^m$? As it turns out $(a^n)^m = a^{nm}$. Consider the example $(a^2)^3$. When we expand this we get $(a^2)^3 = (aa)^3 = (aa)(aa)(aa) = a^6$, just as we should from the claim. Here again, this can be shown pretty easily for integers in general, but also works for real numbers in general.

How should we think about values such as $a^{1/2}$? Since $(a^x)^y = a^{xy}$, we should have that $(a^{1/2})^2 = a^{(1/2)2} = a^1 = a$. From this it follows that $a^{1/2} = \sqrt{a}$. Similarly, $a^{1/n}$ is the nth root of a since if multiplied by itself n times, you must get a.

Logarithmic Functions

The logarithmic function with base a is denoted by $f(x) = \log_a x$. This function is defined only when $x > 0$, $0 < a \neq 1$. It is defined by $y = \log_a x$ if and only if $x = a^y$. The function $\ln x = \log_e x$, where $e \approx 2.71828$, is called the natural log function. When the base is not noted explicitly it is common to treat it as being 10. In other words, if you see $\log a$, it probably means $\log_{10} a$.

Properties of Logs and Exponents

$(a^b)^c = a^{bc}$	$a^b \cdot a^c = a^{b+c}$
When $b \neq 1$, $b^x = b^y$ if and only if $x = y$	
$\log_b b^y = y$	$b^{\log_b c} = c$
$\log_b(MN) = \log_b M + \log_b N$	$\log_b(\frac{M}{N}) = \log_b M - \log_b N$
$\log_b N^y = y \cdot \log_b N$	
$\log_b 1 = 0$ since $b^0 = 1$	$\log_b b = 1$ since $b^1 = b$

Change of Base Formula:

One can change bases using the following formula.

$$\log_a x = \frac{\log_b x}{\log_b a}$$